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THE BUCKLING OF PLATES TAPERED IN PLANFORM

by

G. G. Pope, M.Sc.(Eng)

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1. Plates - Buckling
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SUMMARY

An analysis is given of the buckling of a plate of constant thickness tapered symmetrically in planform and subjected to uniform compressive loading on the parallel ends. Two cases are considered.

(1) Different uniform loads applied normal to the ends, equilibrium being maintained by shear flows along the sides.

(2) Equal uniform stresses applied normal to the ends, with displacement of the sides prevented normal to the direction of taper.

Boundary conditions are such that opposite pairs of edges are either simply-supported or clamped.

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1 INTRODUCTION

In a recent report[1] the author gives an analysis of the buckling of rectangular plates tapered in thickness and loaded in the direction of taper. In this report a similar method is used to analyse the buckling of a plate of constant thickness tapered symmetrically in planform and subjected to uniform compressive loading on the parallel ends. Two cases are considered.

(1) Different uniform normal loads applied to the ends, equilibrium being maintained by shear flows along the sides.

(2) Equal uniform stresses applied normal to the ends with displacement of the sides prevented normal to the direction of taper.

Results are given graphically for plates with boundary conditions such that opposite pairs of edges are either simply-supported or clamped.

The analysis is based on the assumption that the buckled shape normal to the direction of taper differs little from the buckled shape across a rectangular plate of constant thickness under uniform end load with the same boundary conditions along the sides, but simply-supported at the ends. Assuming this transverse buckled form, a linear differential equation with variable coefficients is obtained for the longitudinal deflected shape using the method of Kantorovich[2]. A series solution is derived to this equation.

Klein[3,4] has analysed the buckling of a simply-supported isosceles plate tapered in thickness and in planform by expressing the deflected shape along the axis of symmetry as a Fourier series, and evaluating the coefficients by a collocation method. The results obtained here, which themselves represent an upper limit on the buckling load, are often significantly lower than those calculated by Klein.

2 ASSUMPTIONS

(1) The plate is perfectly elastic.

(2) The transverse buckled shape is the same as that across a rectangular plate of constant thickness under uniform end load with the same boundary conditions along the sides, but simply-supported at the ends.

It is difficult to determine the range of validity of the second assumption, but this analysis should give a good estimate of the buckling load when the sides make an angle of less than, say, 15° with the axis of symmetry.

3 NOTATION (See Fig. 1)

Suffices 1 and 2 on stress, middle-surface force and length symbols indicate values at $x = 0$ and $x = a$ respectively.

a, b	length and width of plate
t	plate thickness
x, y	cartesian co-ordinates
x', y'	local cartesian co-ordinates on side
w	deflection

ρ	$\frac{b_2}{b_1} - 1$
X, Y	$1 + \rho \frac{x}{a}$, $\frac{y}{b_1}$
θ	$\frac{Y}{X}$
W	$\frac{w}{a}$
μ	$\frac{a}{b_1}$
λ	$1 + 0.5\rho$
η	angle made by sides with x axis
E	Young's modulus
ν	Poisson's ratio (taken as 0.3 in computations)
D	flexural rigidity = $E\,t^3/12(1-\nu^2)$
N_x, N_y, N_{xy}	middle surface forces
$\sigma_x, \sigma_y, \tau_{xy}$	middle surface stresses
$\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$	$\frac{\sigma_x}{E}\left(\frac{b_1}{t}\right)^2$ etc.
Δ	$\frac{\bar{\sigma}_{x2}}{\bar{\sigma}_{x1}} - 1$
Ψ	middle surface force function such that $N_x = \frac{\partial^2 \Psi}{\partial y^2}$ etc.
Φ	assumed transverse deflected shape
f	function of X
∇^2	Laplacian differential operator
∇^4	biharmonic differential operator
p, q	defined by equations (9)
$\alpha_i, \beta_i, \gamma$	coefficients defined by equations (1)
ℓ_i, m_i, s_i	coefficients defined by equations (4)

p_i, q_i, r_i	coefficients defined by equations (3)
a_i	coefficients defined by equations (5)
T	work done by middle surface forces
U	strain energy
M_x, M_y	moments referred to x,y axes
$M_{x'}$	moment about side
T'	work done on one side of plate by moment distribution $M_{x'}$

4 GENERAL ANALYSIS

An analysis is given here of the buckling of a plate tapered symmetrically in planform and subjected to different uniform normal loads N_{x1} and N_{x2} along the parallel ends $x = 0$ and $x = a$. The axes and notation used are shown in Fig. 1. The deflection w of the plate can be represented approximately in the form

$$W = \frac{w}{a} = f(X)\ \Phi(\theta)$$

where

$$X = 1 + \rho\frac{x}{a}, \qquad Y = \frac{y}{b_1},$$

$$\rho = \frac{b_2}{b_1} - 1, \qquad \theta = \frac{Y}{X}$$

and b_1 and b_2 are the widths of the plate at $x = 0$ and $x = a$ respectively. The function $\Phi(\theta)$ represents an assumed deflected shape normal to the x axis and the function $f(X)$ can be found by the method of Kantorovich.

The most general system of middle surface forces considered here can be expressed as

$$\left.\begin{aligned} \bar{\sigma}_x &= \alpha_o + \alpha_1 X, \\ \bar{\sigma}_y &= \beta_o + \beta_1 X, \\ \bar{\tau}_{xy} &= \gamma Y \end{aligned}\right\} \tag{1}$$

where $\bar{\sigma}_x = \dfrac{\sigma_x}{E}\left(\dfrac{b_1}{t}\right)^2 = \dfrac{N_x\, b_1^2}{12(1-\nu^2)D}$ etc.,

and the coefficients α_i, β_i and γ are constants.

It is shown in Appendix 1 that, for this system of middle surface forces, the function f(X) satisfies the following differential equation.

$$p_4X^4f'''' + p_3X^3f''' + (p_2 + q_2X^2 + r_2X^3)X^2f'' + (p_1 + q_1X^2 + r_1X^3)Xf' + (p_o + q_oX^2 + r_oX^3)f = 0 \qquad (2)$$

where $f' = \frac{df}{dx}$ etc. and

$$\left.\begin{aligned}
p_o &= \rho^4(\ell_4 + 12\ell_3 + 36\ell_2 + 24\ell_1) + 2\rho^2\mu^2(s_2 + 6s_1 + 6s_o) + \mu^4 m_o , \\
p_1 &= -4\rho^4(\ell_3 + 6\ell_2 + 6\ell_1) - 4\rho^2\mu^2(s_1 + 2s_o) , \\
p_2 &= 6\rho^4(\ell_2 + 2\ell_1) + 2\rho^2\mu^2 s_o , \\
p_3 &= -4\rho^4\ell_1 , \\
p_4 &= \rho^4\ell_o , \\
q_o &= -12(1-\nu^2)\{\rho^2\alpha_o(\ell_2 + 2\ell_1) + \mu^2\beta_o s_o\} , \\
q_1 &= 24(1-\nu^2)\rho^2\alpha_o\ell_1 , \\
q_2 &= -12(1-\nu^2)\rho^2\alpha_o\ell_o , \\
r_o &= -12(1-\nu^2)\left\{\rho^2\alpha_1(\ell_2 + 2\ell_1) + \mu^2\beta_1 s_o - 2\gamma\rho\mu(\ell_1 + \ell_2)\right\} , \\
r_1 &= 24(1-\nu^2)\rho\ell_1(\rho\alpha_1 - \mu\gamma) , \\
r_2 &= -12(1-\nu^2)\rho^2\alpha_1\ell_o
\end{aligned}\right\} \qquad (3)$$

where

$$\mu = \frac{a}{b_1} ,$$

$$\left.\begin{aligned} \ell_i &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \theta^i \Phi \frac{d^i\Phi}{d\theta^i}\, d\theta , & \ell_o &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \Phi^2 d\theta , \\ s_i &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \theta^i \Phi \frac{d^{i+2}\Phi}{d\theta^{i+2}}\, d\theta , & m_o &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \Phi \frac{d^4\Phi}{d\theta^4}\, d\theta . \end{aligned}\right\} \qquad (4)$$

Equation (2) can be solved by expanding f as a power series. To improve the convergence of the series at the ends of the plate at $X = 1$ and $X = 1 + \rho$, the expansion is performed about $X = 1 + 0.5\rho$ by substituting

$$f = Z^c \sum_{r=0}^{\infty} a_n Z^n$$

where

$$Z = X - \lambda$$

and

$$\lambda = 1 + 0.5\rho .$$

The index c is found by equating the coefficient of Z^{c-4} in equation (2) to zero, giving the indicial equation

$$c(c-1)(c-2)(c-3) = 0 .$$

Thus, as equation (2) is linear, the required complete solution is

$$f = \sum_{r=0}^{\infty} a_n Z^n \qquad (5)$$

where the coefficients a_o, a_1, a_2 and a_3 are arbitrary. In general a coefficient a_{n+4} is obtained by equating the coefficient of Z^n in equation (2) to zero, giving

$$
\begin{aligned}
&[r_o + (n-3)r_1 + (n-3)(n-4)r_2]a_{n-3} \\
&+ [q_o + 3\lambda r_o + (n-2)(q_1 + 4\lambda r_1) + (n-2)(n-3)(q_2 + 5\lambda r_2)]a_{n-2} \\
&+ [2q_o + 3\lambda r_o + 3(n-1)(q_1 + 2\lambda r_1) + (n-1)(n-2)(4q_2 + 10\lambda r_2)]\lambda a_{n-1} \\
&+ [p_o + \lambda^2 q_o + \lambda^3 r_o + n(p_1 + 3\lambda^2 q_1 + 4\lambda^3 r_1) + n(n-1)(p_2 + 6\lambda^2 q_2 + 10\lambda^3 r_2) \\
&+ n(n-1)(n-2)p_3 + n(n-1)(n-2)(n-3)p_4]a_n + (n+1)[p_1 + \lambda^2 q_1 + \lambda^3 r_1 \\
&+ n(2p_2 + 4\lambda^2 q_2 + 5\lambda^3 r_2) + 3n(n-1)p_3 + 4n(n-1)(n-2)p_4]\lambda a_{n+1} \\
&+ (n+2)(n+1)[p_2 + \lambda^2 q_2 + \lambda^3 r_2 + 3np_3 + 6n(n-1)p_4]\lambda^2 a_{n+2} \\
&+ (n+3)(n+2)(n+1)[p_3 + 4np_4]\lambda^3 a_{n+3} + (n+4)(n+3)(n+2)(n+1)\lambda^4 a_{n+4} \\
&= 0 . \qquad (6)
\end{aligned}
$$

Coefficients with negative suffices which occur in equation (6) when n is less than 3 are zero by definition.

If the stress coefficients $\bar{\sigma}_{y1}$ and $\bar{\tau}_{xy1}$ are assumed to be proportional to $\bar{\sigma}_{x1}$, the latter can be used as the buckling coefficient. This is evaluated using a digital computer. A value of $\bar{\sigma}_{x1}$ is first assumed which is known to be numerically less than the correct solution, the coefficients a_n of the series are then calculated in terms of the arbitrary constants a_o, a_1, a_2 and a_3 using equation (6). Four linear simultaneous equations are obtained for these constants from the boundary conditions along the ends. The buckling condition is satisfied only if the determinant of the coefficients of these equations is zero. This determinant is evaluated for the assumed value of $\bar{\sigma}_{x1}$, which is then adjusted until the determinant changes sign. Subsequent approximations to $\bar{\sigma}_{x1}$ are made by interpolating and re-evaluating the determinant.

5 MIDDLE SURFACE FORCES

The most general system of middle surface forces considered here is made up of a linear variation of N_x along the plate with a consistent N_{xy} distribution. In this section expressions are derived for N_y under the prescribed conditions on the sides of the plate.

N_x, N_y and N_{xy} are related by the middle surface force function

$$\Psi = N_{x1} \frac{y^2}{2} + (N_{x2} - N_{x1}) \frac{xy^2}{2a} + g(x)$$

so that

$$\left.\begin{aligned} N_x &= N_{x1} + (N_{x2} - N_{x1}) \frac{x}{a} \\ &= N_{x1} \left(1 + \frac{1}{\rho}\right) - \frac{N_{x2}}{\rho} + (N_{x2} - N_{x1}) \frac{X}{\rho}, \\ N_y &= g''(X), \\ N_{xy} &= -(N_{x2} - N_{x1}) \frac{y}{a} \\ &= -(N_{x2} - N_{x1}) \frac{Y}{\mu}. \end{aligned}\right\} \tag{7}$$

If the sides make an angle η with the x axis as shown in Fig. 1, the normal and shear forces $N_{y'}$ and $N_{x'y'}$ along them are given by

$$N_{y'} = N_x \sin^2\eta + N_y \cos^2\eta + N_{xy} \sin 2\eta \tag{8}$$

$$N_{x'y'} = N_{xy} \cos 2\eta - \tfrac{1}{2}(N_y - N_x)\sin 2\eta .$$

Two kinds of loading are considered here:-

(1) No load normal to the sides.

N_y is obtained in terms of N_{x1} and N_{x2} by substituting equations (7) in equation (8) and putting $N_{y'} = 0$. Expressing $\tan\eta$ in terms of ρ and μ, this gives

$$N_y = -\frac{\rho^2}{4\mu^2} \left\{N_{x1} \left(1 + \frac{1}{\rho}\right) - \frac{N_{x2}}{\rho}\right\} - \frac{3\rho X}{4\mu^2} (N_{x2} - N_{x1}).$$

Thus the loading can be expressed in the notation of equations (1) as

$$\alpha_o = \bar{\sigma}_{x1}\mu^2\left(1 - \frac{\Delta}{\rho}\right), \qquad \alpha_1 = \bar{\sigma}_{x1}\frac{\mu^2\Delta}{\rho},$$

$$\beta_o = -\bar{\sigma}_{x1}\frac{\rho^2}{4}\left(1 - \frac{\Delta}{\rho}\right), \qquad \beta_1 = -\frac{3\bar{\sigma}_{x1}}{4}\rho\Delta,$$

$$\gamma = -\bar{\sigma}_{x1}\mu\Delta$$

where

$$\Delta = \frac{N_{x2}}{N_{x1}} - 1 .$$

(2) Constant longitudinal stress and no displacement of the sides normal to the direction of taper.

Here

$$N_{x1} = N_{x2}, \qquad \alpha_o = \bar{\sigma}_{x1}\mu^2,$$

$$N_y = \nu N_x, \qquad \beta_o = \nu\bar{\sigma}_{x1}\mu^2,$$

$$N_{xy} = 0, \qquad \alpha_1 = \beta_1 = \gamma = 0 .$$

6 APPLICATIONS

The preceding analysis has been computed for plates with boundary conditions such that opposite pairs of edges are either simply-supported or clamped. Expressions are given in sections 6.1 and 6.2 for the constants required in equation (2) under these conditions and a list of the results plotted is given in section 6.3.

6.1 Plate simply-supported along sides

The assumed transverse buckled form, which is here given by

$$\Phi = \cos \pi\theta ,$$

does not completely satisfy the condition of simple support along the sides of the plate. It is shown in Appendix 2 however that the total work done on the plate by the spurious moments introduced along the sides is zero. The analysis given in Appendix 1 thus remains valid.

The constants required in equation (2) are here

$$\ell_0 = \frac{1}{2}, \qquad m_0 = \frac{\pi^4}{2},$$

$$\ell_1 = -\frac{1}{4}, \qquad s_0 = -\frac{\pi^2}{2}$$

$$\ell_2 = -\frac{1}{24}(\pi^2 - 6), \qquad s_1 = \frac{\pi^2}{4}$$

$$\ell_3 = \frac{1}{16}(\pi^2 - 6), \qquad s_2 = \frac{\pi^2}{24}(\pi^2 - 6).$$

$$\ell_4 = \frac{1}{160}(\pi^4 - 20\pi^2 + 120),$$

6.2 Plate clamped along sides

The assumed transverse buckled form is here given by

$$\Phi = \cosh p\theta - q \cos p\theta$$

where p is the first positive root (4.73004) of the equation

$$\left.\begin{aligned} \sinh\frac{p}{2}\cos\frac{p}{2} + \cosh\frac{p}{2}\sin\frac{p}{2} &= 0 \\ \text{and} \qquad q &= \frac{\cosh\frac{p}{2}}{\cos\frac{p}{2}}. \\ &= (\sec p)^{\frac{1}{2}} \end{aligned}\right\} \qquad (9)$$

This deflected shape satisfies completely the boundary conditions along the sides.

The constants required in equation (2) are given by

$$\ell_o = \frac{1}{2}(q^2+1) ,$$

$$\ell_1 = -\frac{1}{4}(q^2+1) ,$$

$$\ell_2 = -\frac{p^2}{24}(q^2-1) - \frac{p}{4}(q^4-1)^{\frac{1}{2}} - \frac{1}{4}(q^2+1) ,$$

$$\ell_3 = \frac{3p^2}{16}(q^2-1) - \frac{3p}{4}(q^4-1)^{\frac{1}{2}} + \frac{15}{8}(q^2+1) ,$$

$$\ell_4 = \frac{p^4}{160}(q^2+1) - \frac{5p^2}{8}(q^2-1) - \frac{15p}{4}(q^4-1)^{\frac{1}{2}} - \frac{27p}{4}(q^2+1) ,$$

$$m_o = p^4\ell_o ,$$

$$\frac{s_o}{p} = -\frac{p}{2}(q^2-1) - (q^4-1)^{\frac{1}{2}} ,$$

$$\frac{s_1}{p} = \frac{3p}{4}(q^2-1) + \frac{3}{2}(q^4-1)^{\frac{1}{2}} ,$$

$$\frac{s_2}{p} = \frac{p^2}{24}(q^2+1) - \frac{5p}{4}(q^2-1) - \frac{5}{2}(q^4-1)^{\frac{1}{2}} .$$

6.3 Results

The variation of the buckling coefficient $\bar{\sigma}_{x1}$ with $\frac{a}{b_1}$ is plotted for a series of values of $\frac{b_2}{b_1}$ and $\frac{N_{x2}}{N_{x1}}$ in the figures listed below with various combinations of the boundary conditions. The corresponding curves for rectangular plates, which are also given in these figures, have been obtained where possible from R.Ae.S. data sheets; other cases have been calculated by a method similar to that given in this Report.

(a) Buckling coefficient of plate with no normal load along sides.

End boundary conditions	Side boundary conditions	$\frac{N_{x2}}{N_{x1}}$	Fig.
simply-supported	simply-supported	0.8	2
"	"	1	3
"	"	1.2	4
clamped	"	0.8	5
"	"	1	6
"	"	1.2	7
simply-supported	clamped	0.8	8
"	"	1	9
"	"	1.2	10
clamped	"	0.8	11
"	"	1	12
"	"	1.2	13

(b) Buckling coefficient of plate with no displacement of the sides normal to the direction of taper.

End boundary conditions	Side boundary conditions	$\frac{N_{x2}}{N_{x1}}$	Fig.
simply-supported	simply-supported	1	14
clamped	"	1	15
simply-supported	clamped	1	16
clamped	"	1	17

Specimen buckled shapes for the above loadings are shown in Figs. 18 and 19.

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ATTACHED:

Appendices 1 and 2
Figs.1-19, SME 86745/R-86763/R
Detachable abstract cards

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APPENDIX 1

BASIC ANALYSIS OF THE BUCKLING OF AN ISOSCELES TRAPEZOIDAL PLATE OF CONSTANT THICKNESS, USING THE METHOD OF KANTOROVICH

In classical small deflection theory[5], the strain energy of bending of a plate is given by

$$U = \iint \frac{D}{2}\left\{(\nabla^2 w)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dxdy.$$

The increment of the strain energy due to an infinitesimal arbitrary variation δw of the deflection is thus

$$\delta U = \iint D\left\{\nabla^2 w \nabla^2(\delta w) - (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 \delta w}{\partial y^2} + \frac{\partial^2 \delta w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2 w}{\partial x \partial y}\,\frac{\partial^2 \delta w}{\partial x \partial y}\right]\right\} dxdy.$$

Provided the variation δw satisfies the boundary conditions of the plate and the flexural rigidity D is constant, this expression can be integrated by parts twice to give

$$\delta U = D \iint \delta w\, \nabla^4 w \, dx\, dy\ . \tag{9}$$

The work done on the plate by the middle surface forces N_x N_y and N_{xy} is given by

$$T = -\frac{1}{2}\iint\left\{N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\right\} dxdy.$$

Thus the increment of the work done due to the variation δw is given by

$$\delta T = -\iint\left\{N_x\,\frac{\partial w}{\partial x}\,\frac{\partial \delta w}{\partial x} + N_y\,\frac{\partial w}{\partial y}\,\frac{\partial \delta w}{\partial y} + N_{xy}\left(\frac{\partial w}{\partial x}\,\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\,\frac{\partial \delta w}{\partial x}\right)\right\} dxdy.$$

Provided both the deflection w and the variation δw are zero along the edges of the plate and the middle surface forces are in equilibrium, the increment δT of the work done can be integrated by parts to give

$$\delta T = \iint \delta w\left(N_x\,\frac{\partial^2 w}{\partial x^2} + N_y\,\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\,\frac{\partial^2 w}{\partial x \partial y}\right) dxdy\ . \tag{10}$$

Equating expressions (9) and (10), we obtain

$$\iint \delta w \left\{ \nabla^4 w - \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \right\} dxdy = 0$$

which may be rewritten in the non-dimensional form

$$\iint \delta W \left\{ \rho^4 \frac{\partial^4 W}{\partial X^4} + 2\rho^2\mu^2 \frac{\partial^4 W}{\partial X^2 \partial Y^2} + \mu^4 \frac{\partial^4 W}{\partial Y^4} - 12(1-\nu^2)\mu^2 \left(\bar{\sigma}_x \rho^2 \frac{\partial^2 W}{\partial X^2} + \bar{\sigma}_y \mu^2 \frac{\partial^2 W}{\partial Y^2} + 2\bar{\tau}_{xy} \rho\, \mu \frac{\partial^2 W}{\partial X \partial Y} \right) \right\} dXdY = 0 \qquad (11)$$

where

$$W = \frac{w}{a} \qquad X = 1 + \rho\frac{x}{a}, \qquad Y = \frac{y}{b_1},$$

$$\rho = \frac{b_2}{b_1} - 1, \qquad \mu = \frac{a}{b_1},$$

$$\bar{\sigma}_x = \frac{a^2 N_x}{12(1-\nu^2)\mu^2 D} = \sigma_x \left(\frac{b_1}{t}\right)^2, \qquad \bar{\sigma}_y = \sigma_y \left(\frac{b_1}{t}\right)^2 \text{ etc.}$$

and b_1 and b_2 are the widths of the plate at $x = 0$ and $x = a$ as shown in Fig. 1.

If W were the true deflected shape of the plate and the variation δW were completely arbitrary, this integral would yield the differential equation for the deformation of the plate. In the present analysis however W is represented approximately by the expression

$$W = f(X)\, \Phi(\theta)$$

where

$$\theta = \frac{Y}{X}$$

and Φ is an assumed function. The ordinary differential equation for f is found by considering a restricted variation δW such that

$$\delta W = \Phi\, \delta f.$$

As the variation δf is arbitrary, the integration sign with respect to X can be removed from expression (11), leaving

$$\int_{-\frac{X}{2}}^{+\frac{X}{2}} \Phi \left\{ \rho^4 \frac{\partial^4 W}{\partial X^4} + 2\rho^2\mu^2 \frac{\partial^4 W}{\partial X^2 \partial Y^2} + \mu^4 \frac{\partial^4 W}{\partial Y^4} - 12(1-\nu^2)\mu^2 \left(\bar{\sigma}_x \rho^2 \frac{\partial^2 W}{\partial X^2} \right.\right.$$

$$\left.\left. + \bar{\sigma}_y \mu^2 \frac{\partial^2 W}{\partial Y^2} + 2\bar{\tau}_{xy} \rho\mu \frac{\partial^2 W}{\partial X \partial Y} \right) \right\} dY = 0 . \qquad (12)$$

Now the most general system of middle surface forces considered here is

$$\bar{\sigma}_x = \alpha_o + \alpha_1 X ,$$

$$\bar{\sigma}_y = \beta_o + \beta_1 X ,$$

$$\bar{\tau}_{xy} = \gamma Y ,$$

where the coefficients α_i, β_i and γ are constants.

Substituting for W in terms of f and Φ in equation (12) and integrating with respect to Y, the following differential equation is obtained for f.

$$p_4 X^4 f'''' + p_3 X^3 f''' + (p_2 + q_2 X^2 + r_2 X^3) X^2 f'' + (p_1 + q_1 X^2 + r_1 X^3) X f'$$

$$+ (p_o + q_o X^2 + r_o X^3) f = 0 \qquad (2')$$

where the coefficients are defined by equations (3) and (4).

The arbitrary constants in the solution of equation (2) are found from the boundary conditions at the ends of the plate. When the ends are clamped these conditions are satisfied completely; when the ends are simply-supported however the condition of zero moment cannot be satisfied exactly, but is represented approximately by the expression

$$\ell_o f'' - \frac{2}{X} \ell_1 f' = 0$$

which satisfies the requirement that the variation δf should do no work on the boundary.

APPENDIX 2

PLATES WITH SIMPLY-SUPPORTED SIDES

In this report the buckled form of plates simply-supported along the sides has been expressed in the form

$$W = f\Phi$$

where the function Φ satisfies the boundary conditions

$$[\Phi]_{\theta=\pm\frac{1}{2}} = [\Phi'']_{\theta=\pm\frac{1}{2}} = 0, \text{ where } \Phi' = \frac{d\Phi}{d\theta} \text{ etc.}$$

The bending moment distribution $M_{x'}$ about the sides of the plate which would strictly be necessary for the plate to assume this buckled form is given by

$$M_{x'} = \frac{1}{1+\nu}\,[M_x + M_y]_{\theta=\pm\frac{1}{2}} \, . \tag{13}$$

Now the bending moments M_x and M_y are given by

$$M_x = -\frac{D}{a}\left(\rho^2\frac{\partial^2 W}{\partial X^2} + \nu\mu^2\frac{\partial^2 W}{\partial Y^2}\right), \qquad M_y = -\frac{D}{a}\left(\mu^2\frac{\partial^2 W}{\partial Y^2} + \nu\rho^2\frac{\partial^2 W}{\partial X^2}\right) \tag{14}$$

where the derivatives of W can be expressed as

$$\frac{\partial^2 W}{\partial X^2} = \frac{1}{X^2}\,[\Phi X^2 f'' - 2\theta\Phi' X f' + \theta(\theta\Phi'' + 2\Phi')f]\,, \qquad \frac{\partial^2 W}{\partial Y^2} = \frac{1}{X^2}\,\Phi'' f \, . \tag{15}$$

Substituting equations (14) and (15) in equation (13), we obtain

$$M_{x'} = \frac{\rho^2 D\Phi'}{(1+\nu)aX^2}\,(Xf' - f) \, . \tag{16}$$

Now the slope of the plate normal to the side $\theta = \frac{1}{2}$ is given by

$$\left[\frac{\partial w}{\partial y'}\right]_{\theta=\frac{1}{2}} = \mu\frac{\partial W}{\partial Y}\cos\eta + \rho\frac{\partial W}{\partial X}\sin\eta$$

$$= \left(\mu\cos\eta - \frac{\rho}{2}\sin\eta\right)\frac{f\Phi'}{X} \, . \tag{17}$$

Hence from equations (16) and (17) the total work done on the side $\theta = \frac{1}{2}$ by the moment $M_{x'}$ is given by

$$T' = \frac{\rho^2 D\Phi'^2}{2(1+\nu)a}\left(\mu \cos \eta - \frac{\rho}{2} \sin \eta\right)\int_1^{1+\rho} \frac{f}{X^3}(Xf' - f)dX$$

$$= \frac{\rho^2 D\Phi'^2}{4(1+\nu)a}\left(\mu \cos \eta - \frac{\rho}{2} \sin \eta\right)\left[\frac{f^2}{X^2}\right]_1^{1+\rho}.$$

Now the deflection at the ends of the plates considered in this report is zero, hence

$$T' = 0.$$

Similarly integrating $M_{x'}$ along the side $\theta = \frac{1}{2}$, it can be shown that the moments along each side are self-equilibrating.

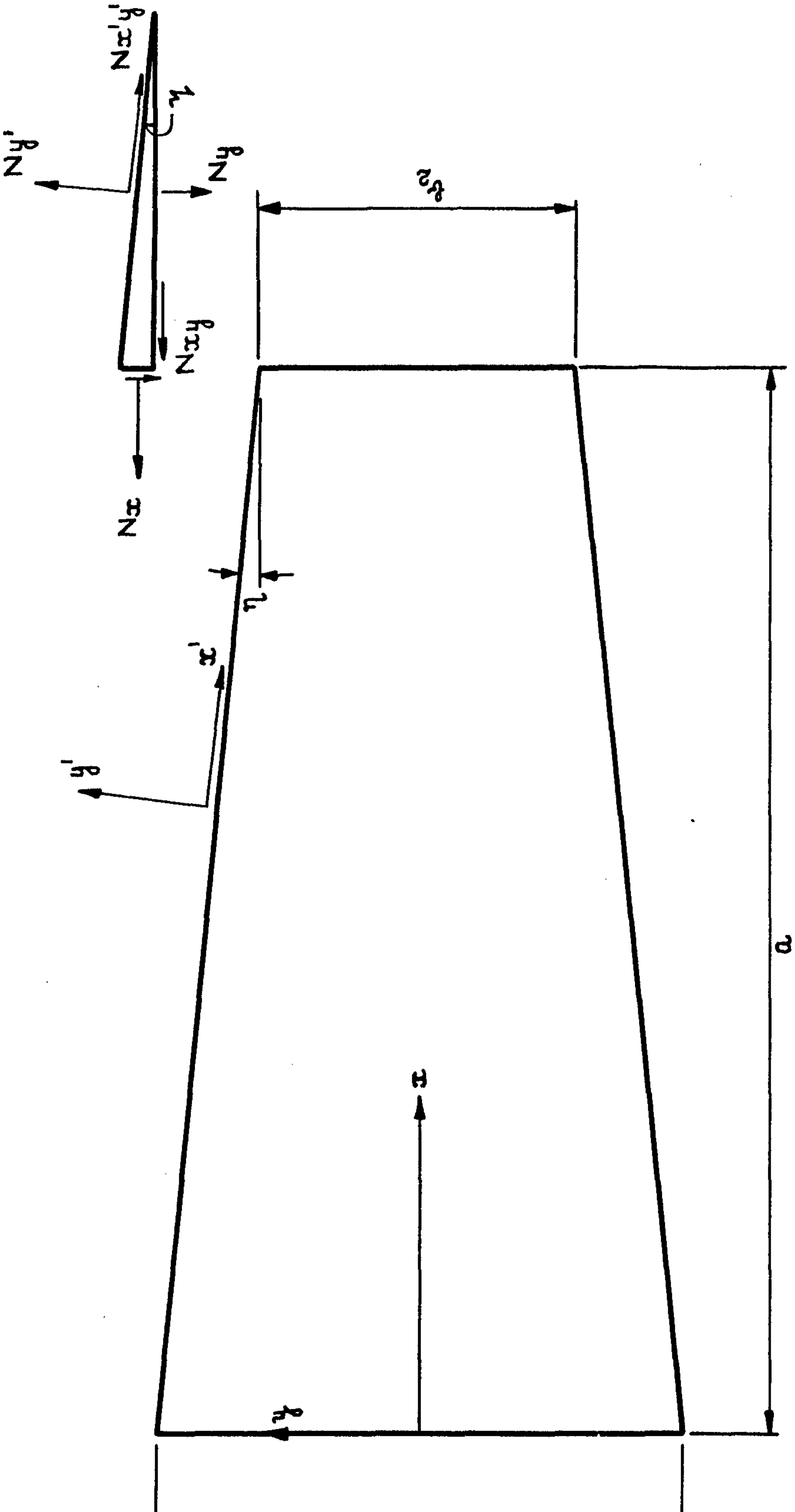


FIG. 1. AXES AND NOTATION.

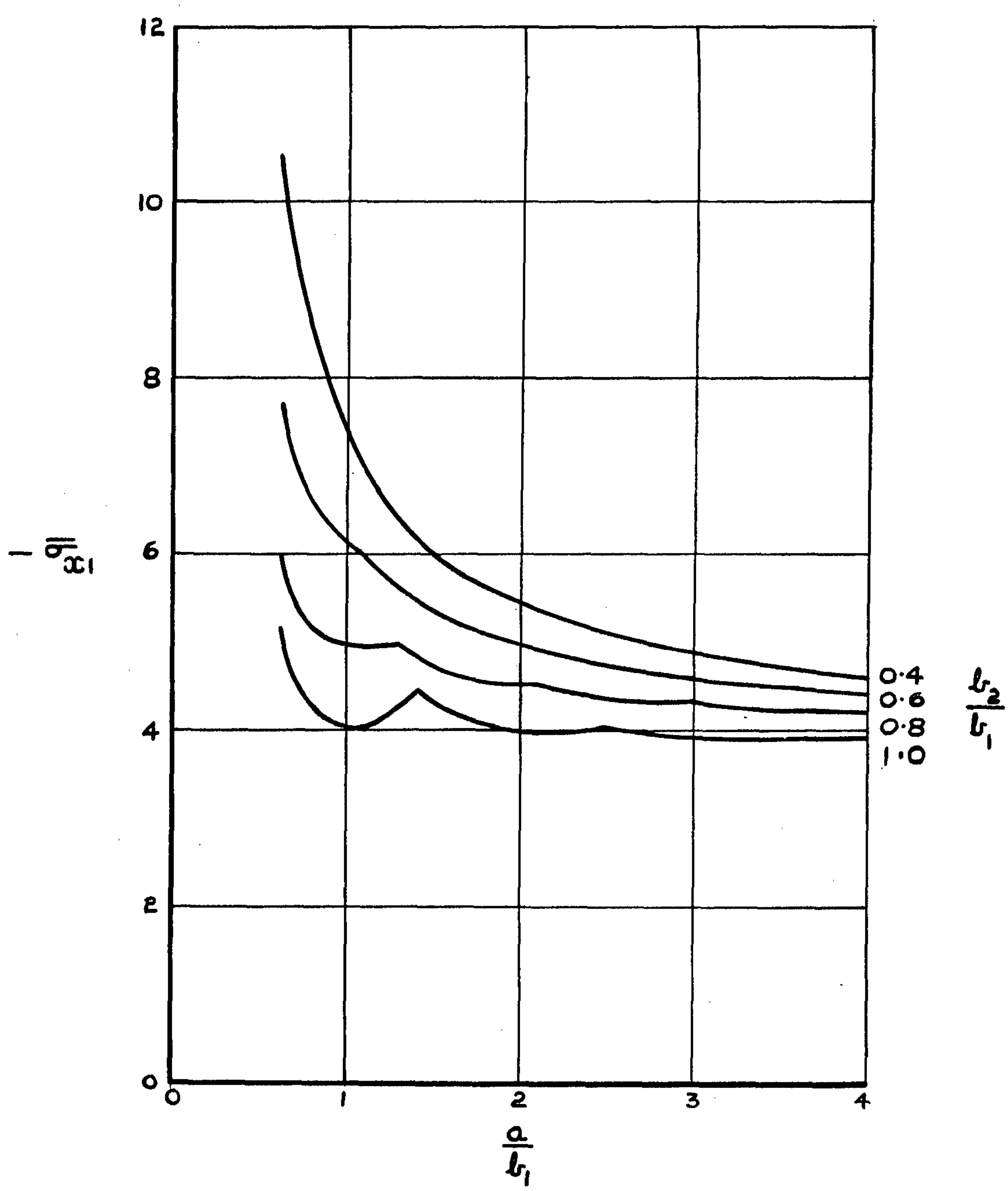


FIG. 2. BUCKLING STRESS DIAGRAM. SIDES AND ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 0{\cdot}8$.

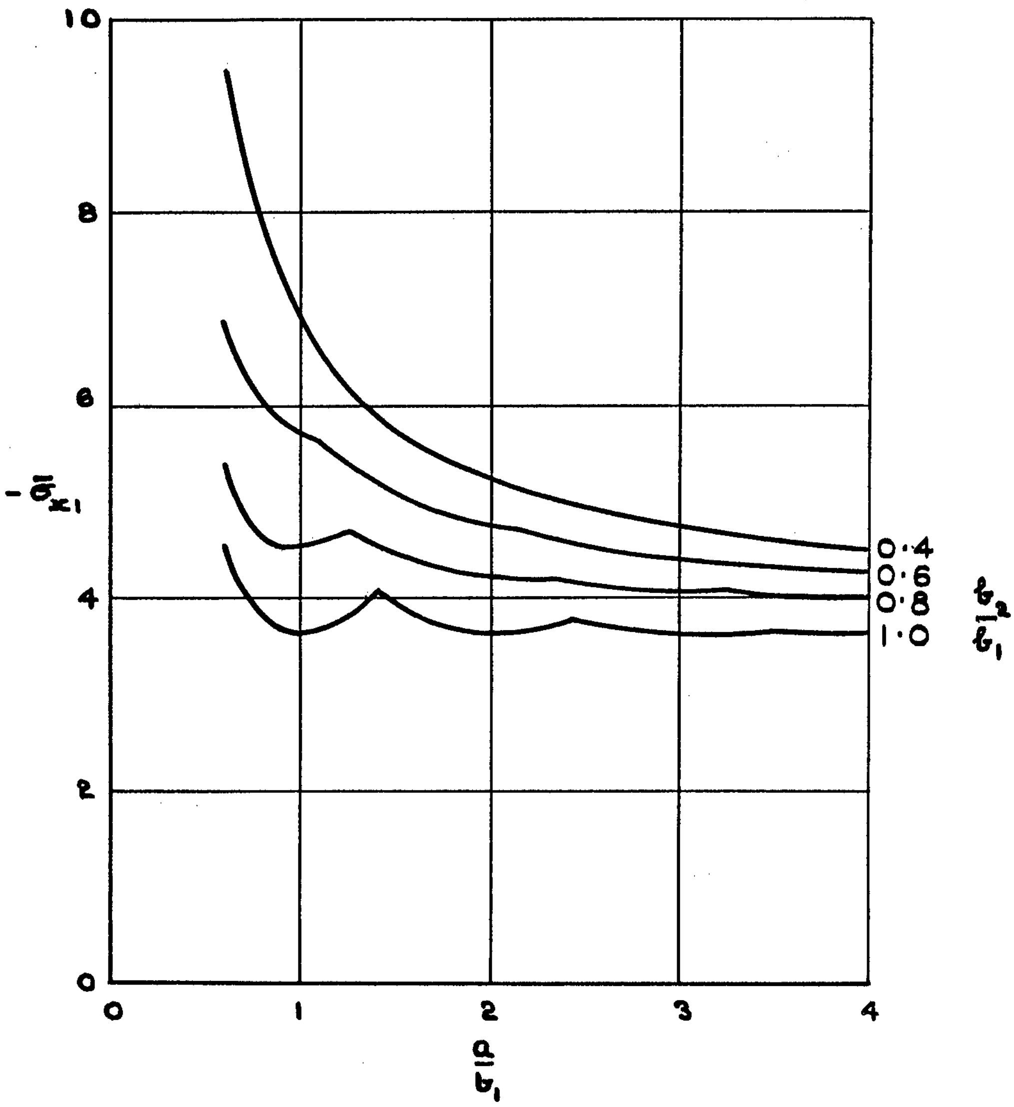


FIG. 3. BUCKLING STRESS DIAGRAM. SIDES AND ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES.

$N_{x2}/N_{x1} = 1.$

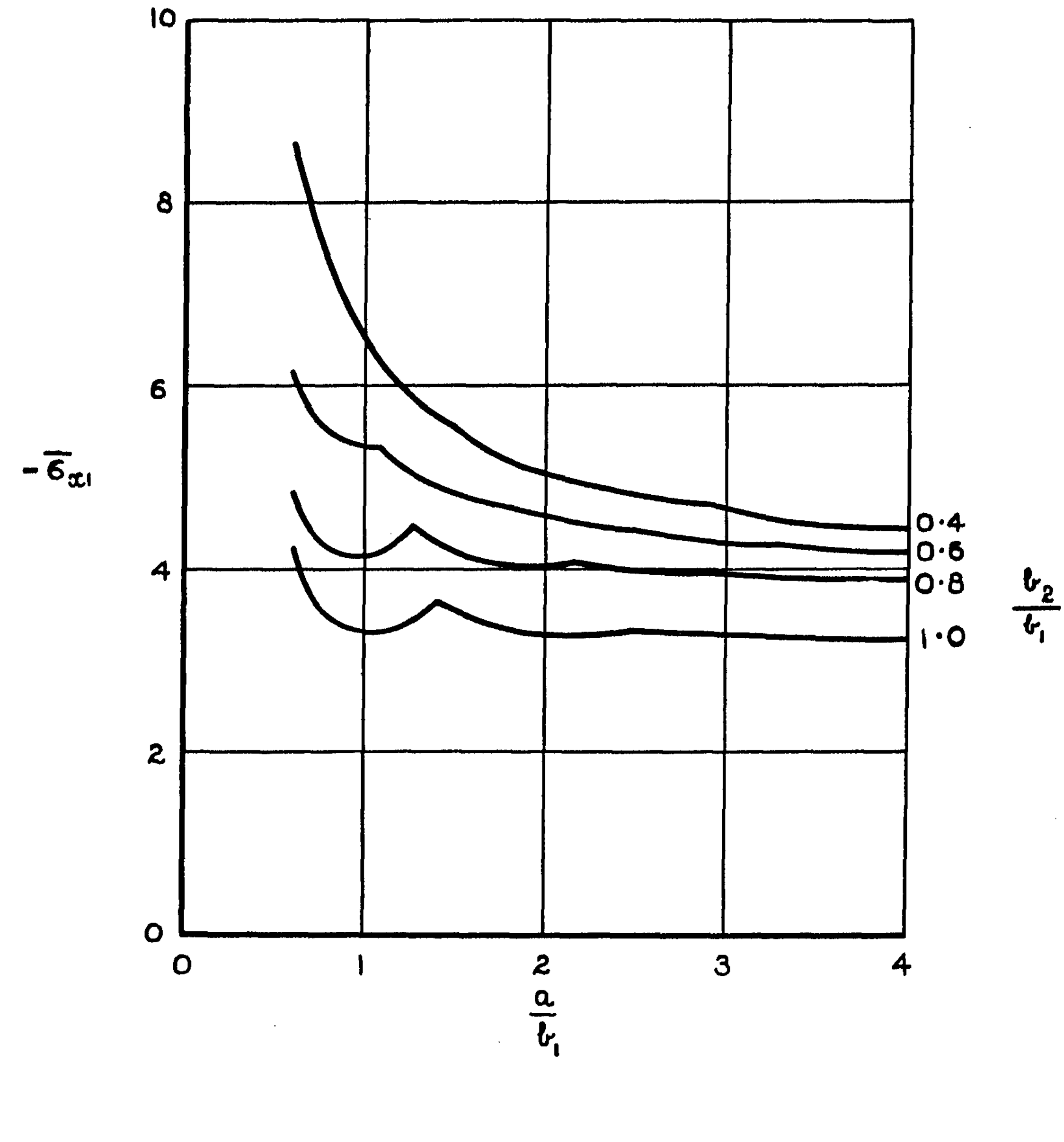


FIG. 4. BUCKLING STRESS DIAGRAM. SIDES AND ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 1{\cdot}2$.

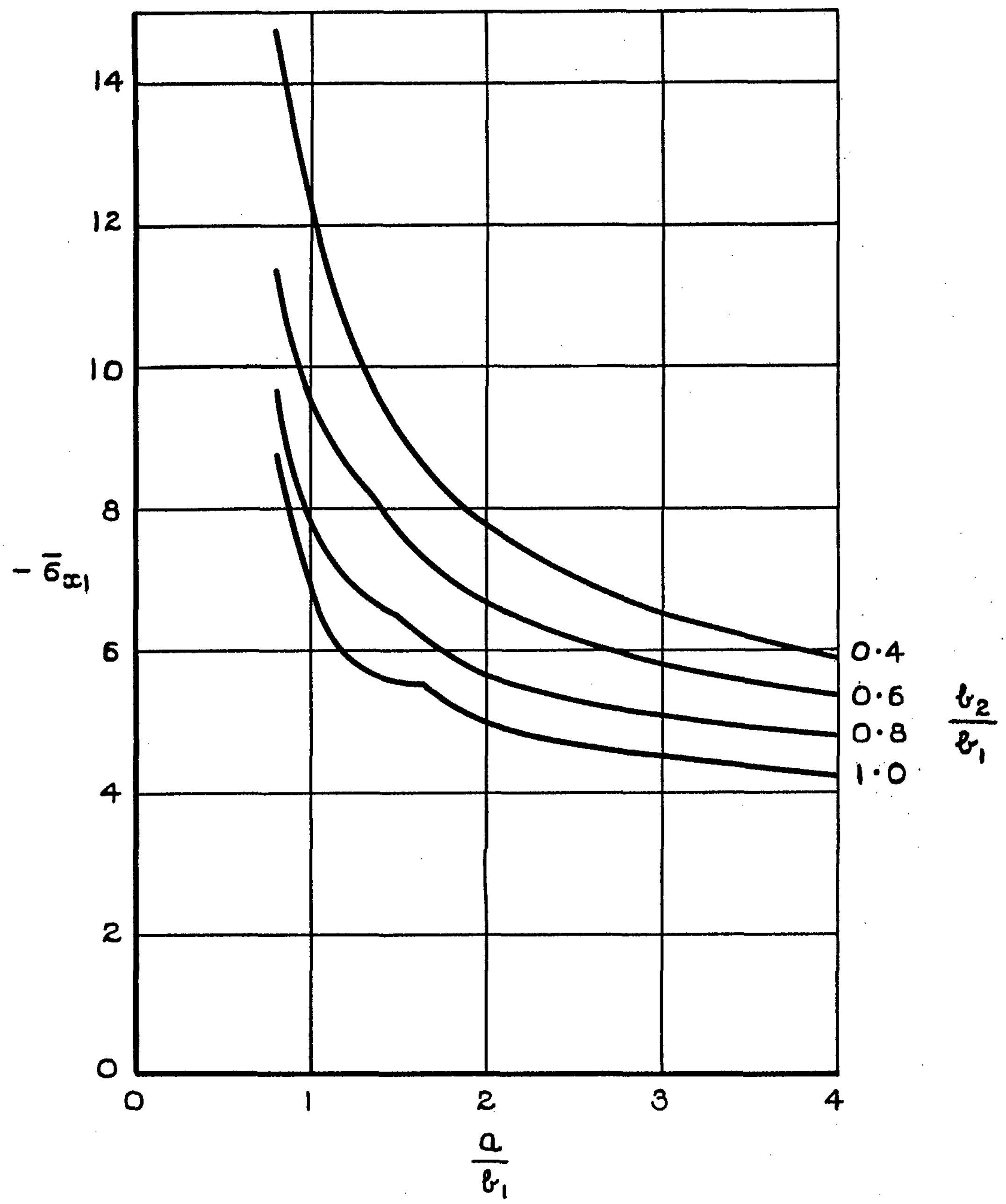


FIG. 5. BUCKLING STRESS DIAGRAM. SIDES SIMPLY-SUPPORTED. ENDS CLAMPED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 0\cdot 8$.

FIG. 6.

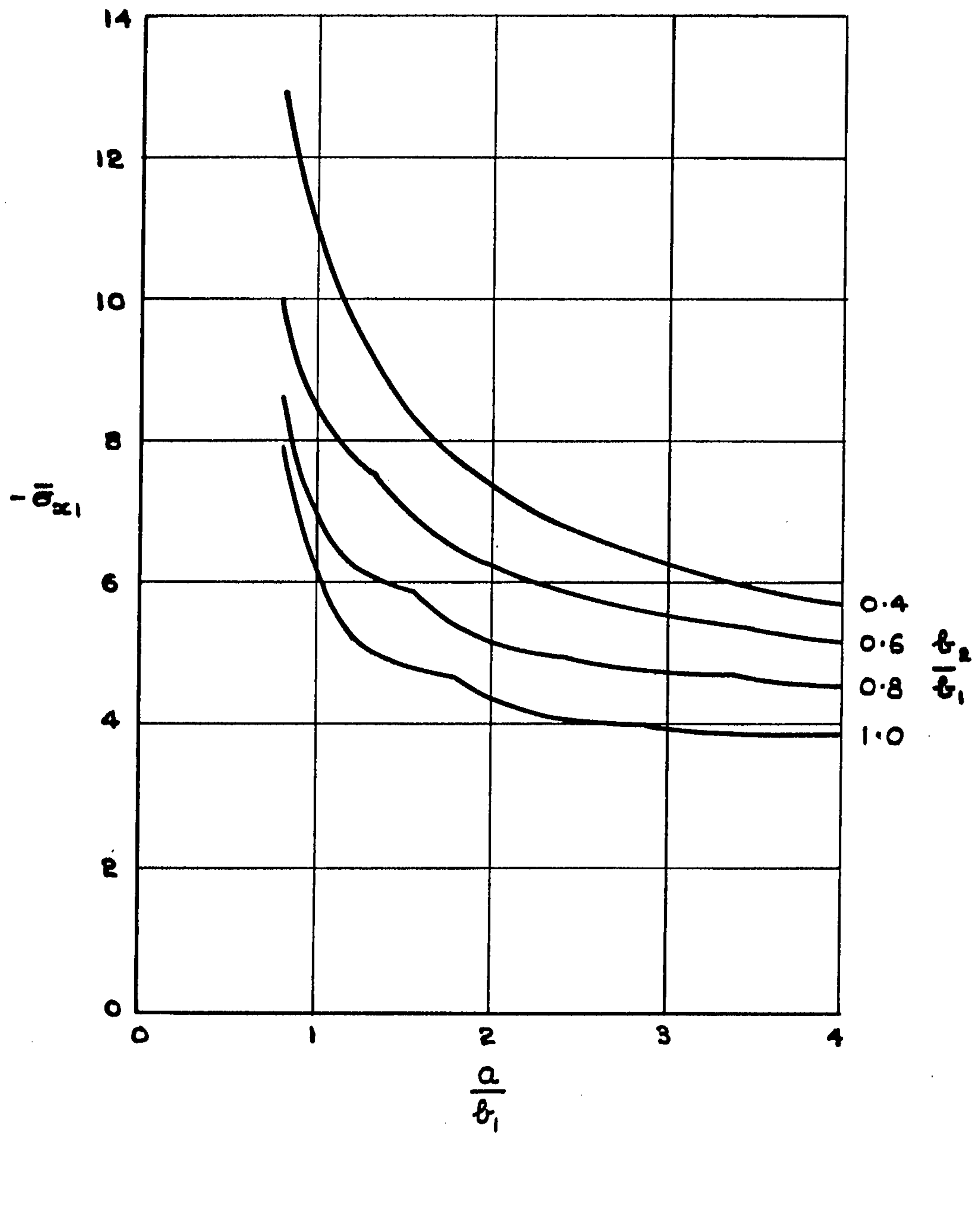


FIG. 6. BUCKLING STRESS DIAGRAM. SIDES SIMPLY-SUPPORTED. ENDS CLAMPED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 1 \cdot 0$.

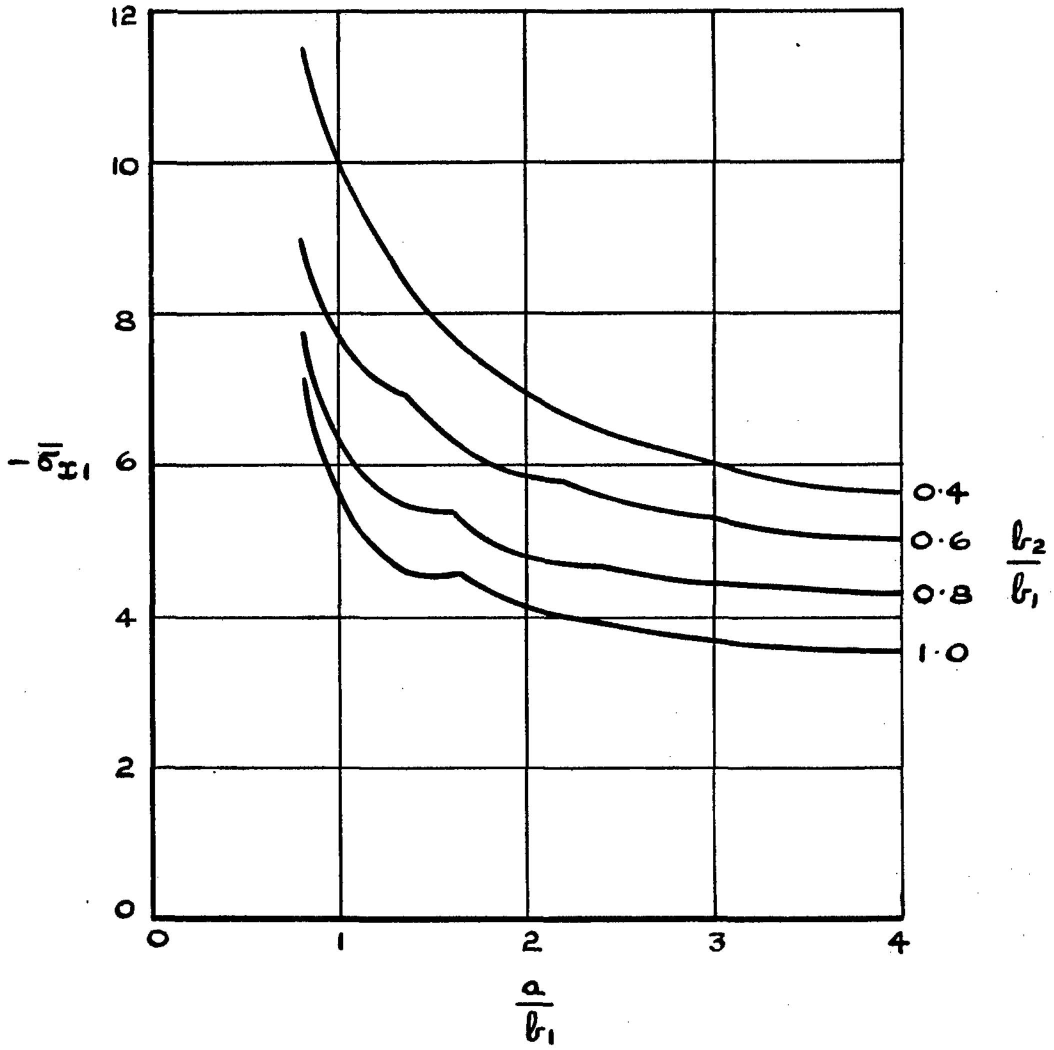


FIG. 7. BUCKLING STRESS DIAGRAM. SIDES SIMPLY-SUPPORTED. ENDS CLAMPED. NO STRESS NORMAL TO SIDES. $N_{x2} / N_{x1} = 1 \cdot 2$.

FIG. 8.

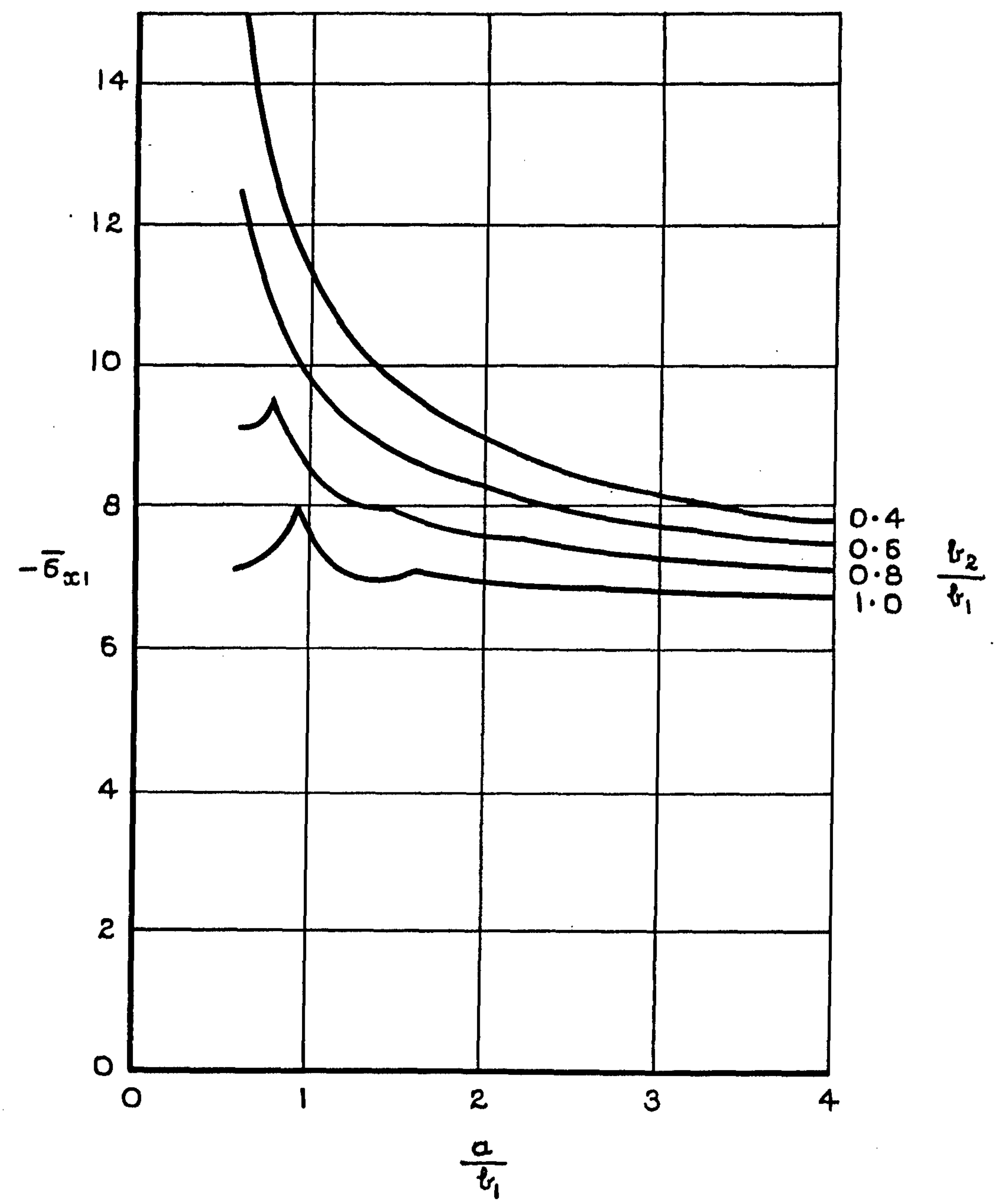


FIG. 8. BUCKLING STRESS DIAGRAM. SIDES CLAMPED. ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 0{\cdot}8$.

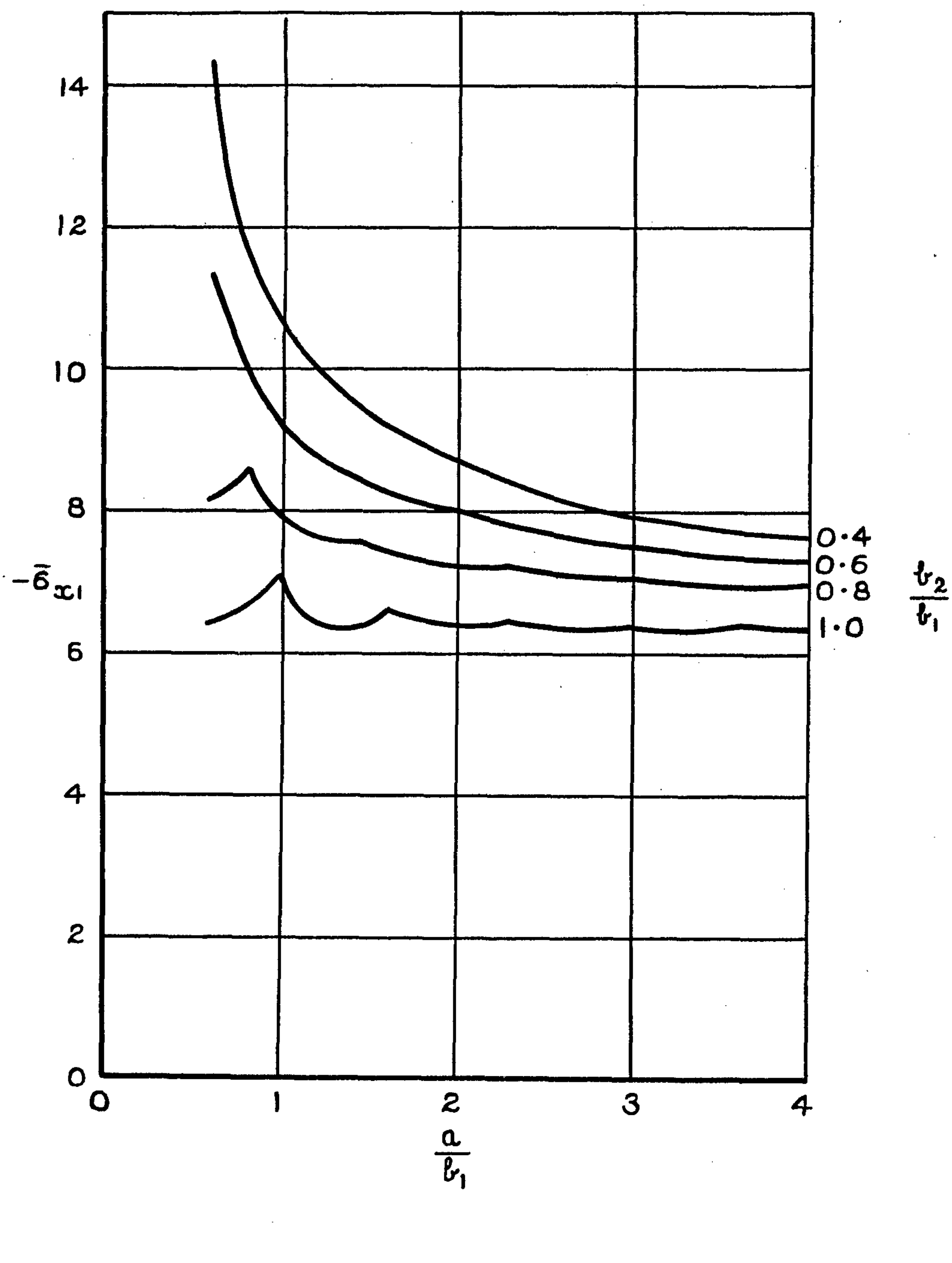


FIG. 9. BUCKLING STRESS DIAGRAM. SIDES CLAMPED. ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 1 \cdot 0$.

FIG. 10.

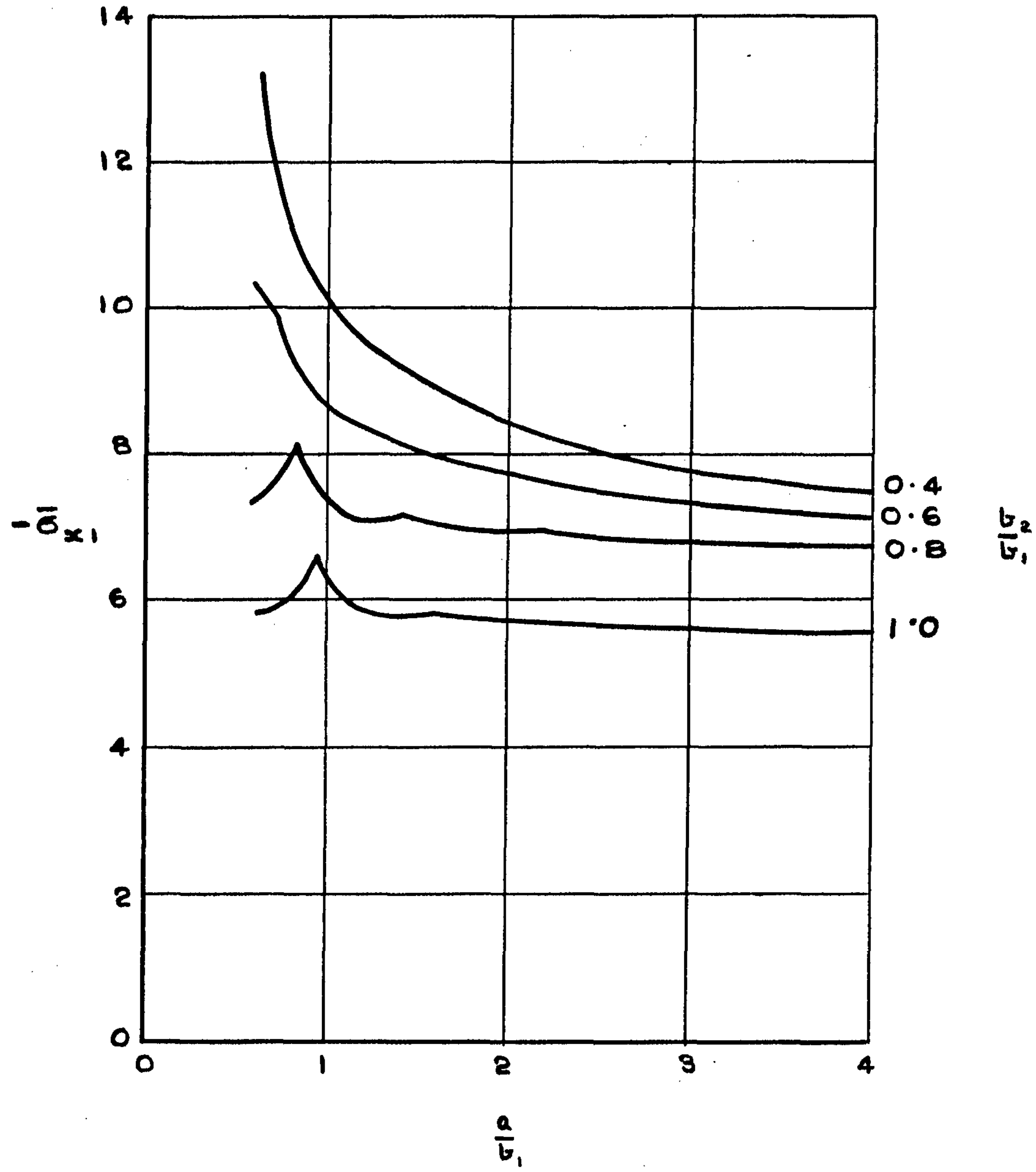


FIG. 10. BUCKLING STRESS DIAGRAM. SIDES CLAMPED. ENDS SIMPLY-SUPPORTED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 1 \cdot 2$.

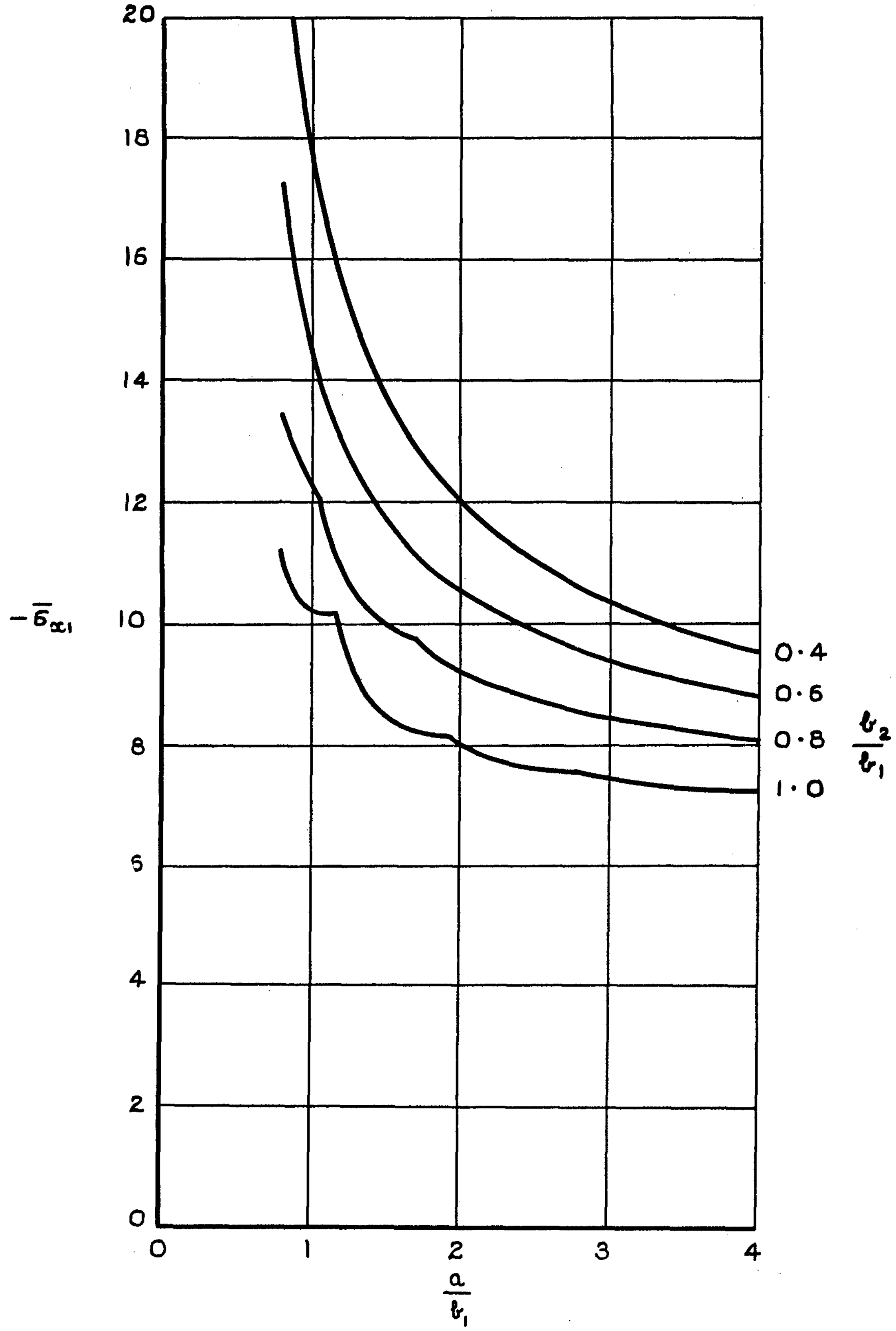


FIG. 11. BUCKLING STRESS DIAGRAM. SIDES AND ENDS CLAMPED. NO STRESS NORMAL TO SIDES.

$N_{x2}/N_{x1} = 0{\cdot}8.$

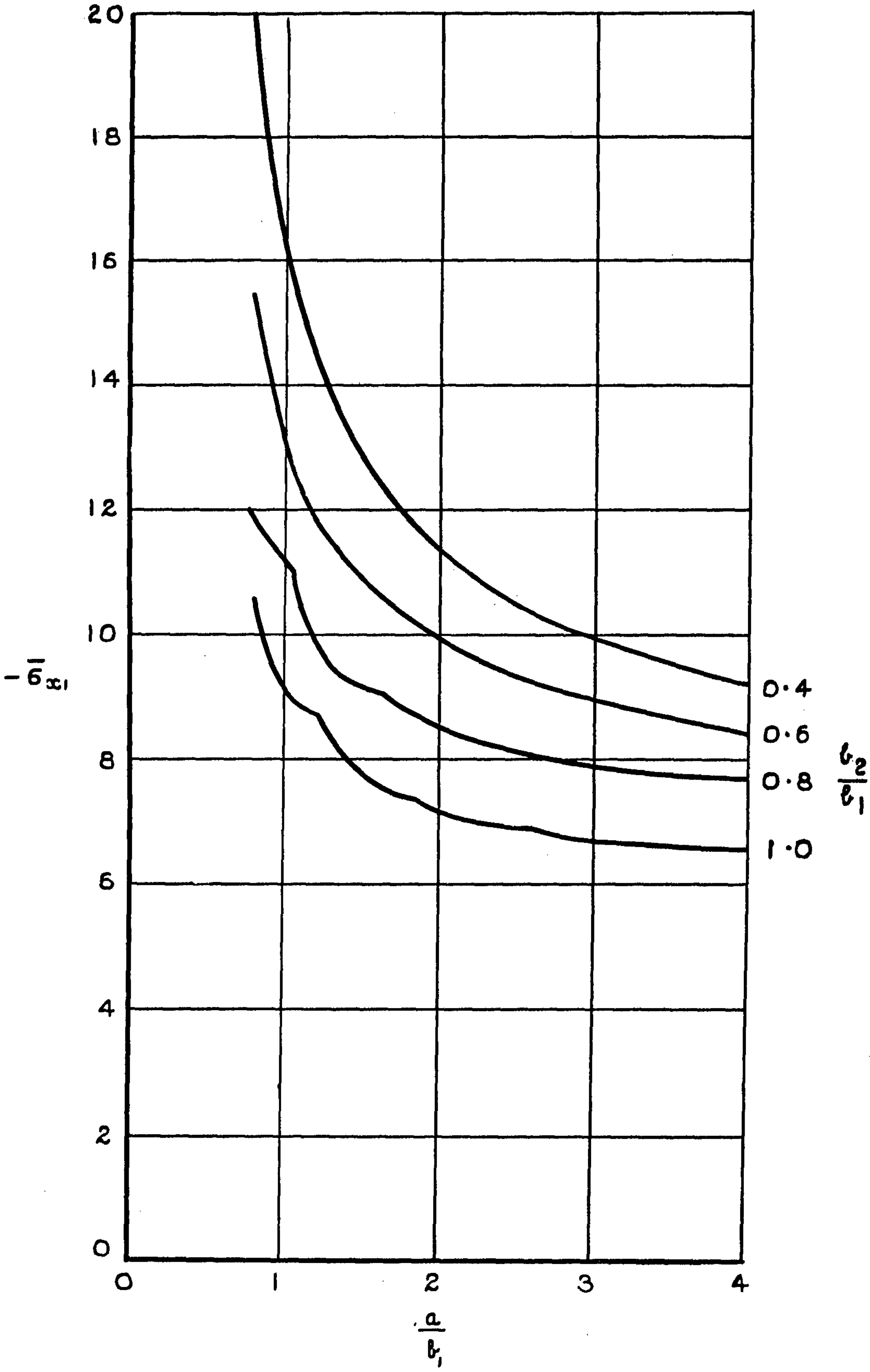


FIG. 12. BUCKLING STRESS DIAGRAM. SIDES AND ENDS CLAMPED. NO STRESS NORMAL TO SIDES. $N_{x2}/N_{x1} = 1$.

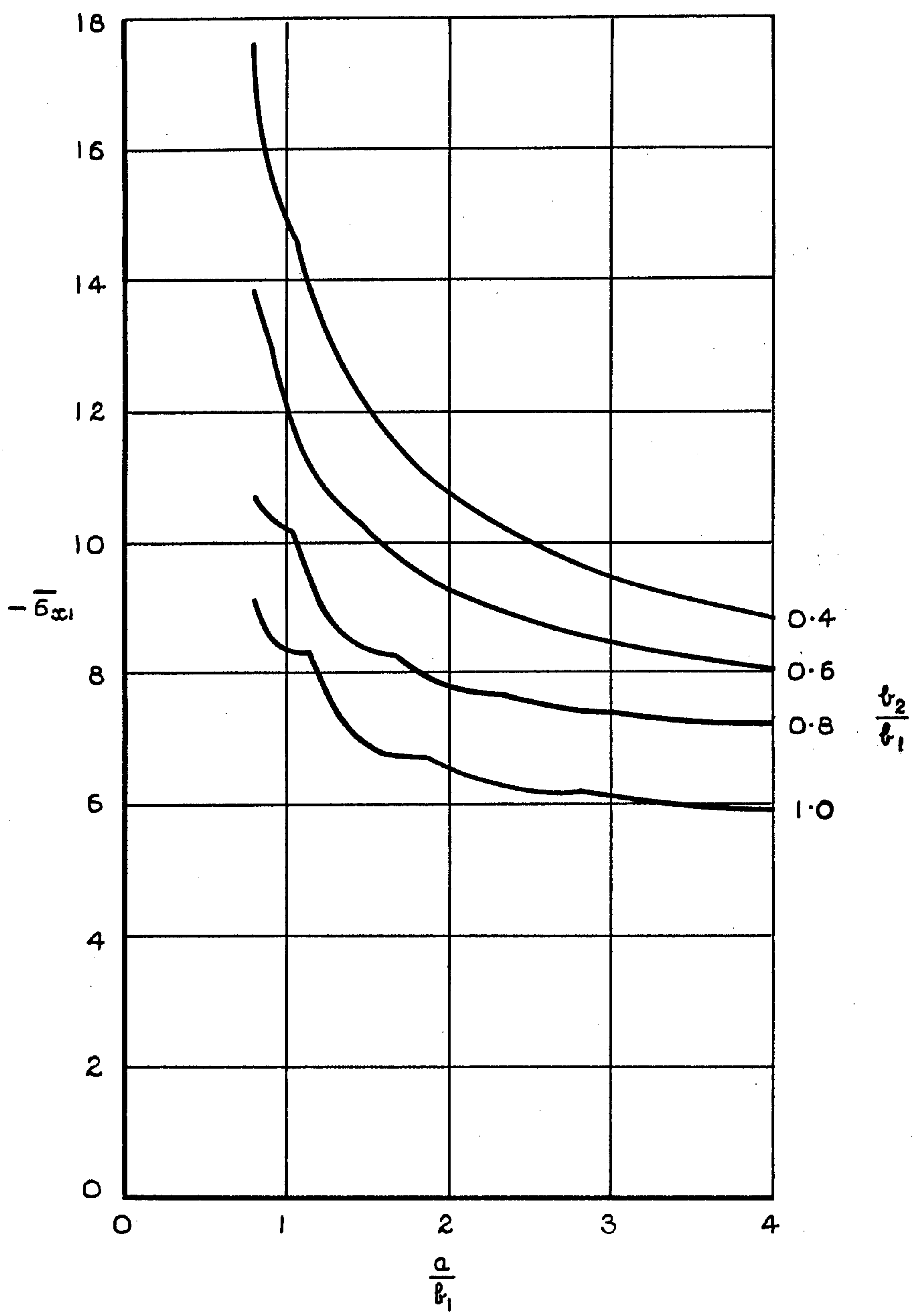


FIG.13. BUCKLING STRESS DIAGRAM. SIDES AND ENDS CLAMPED. NO STRESS NORMAL TO SIDES.

$N_{x2}/N_{x1} = 1\cdot2.$

FIG.14.

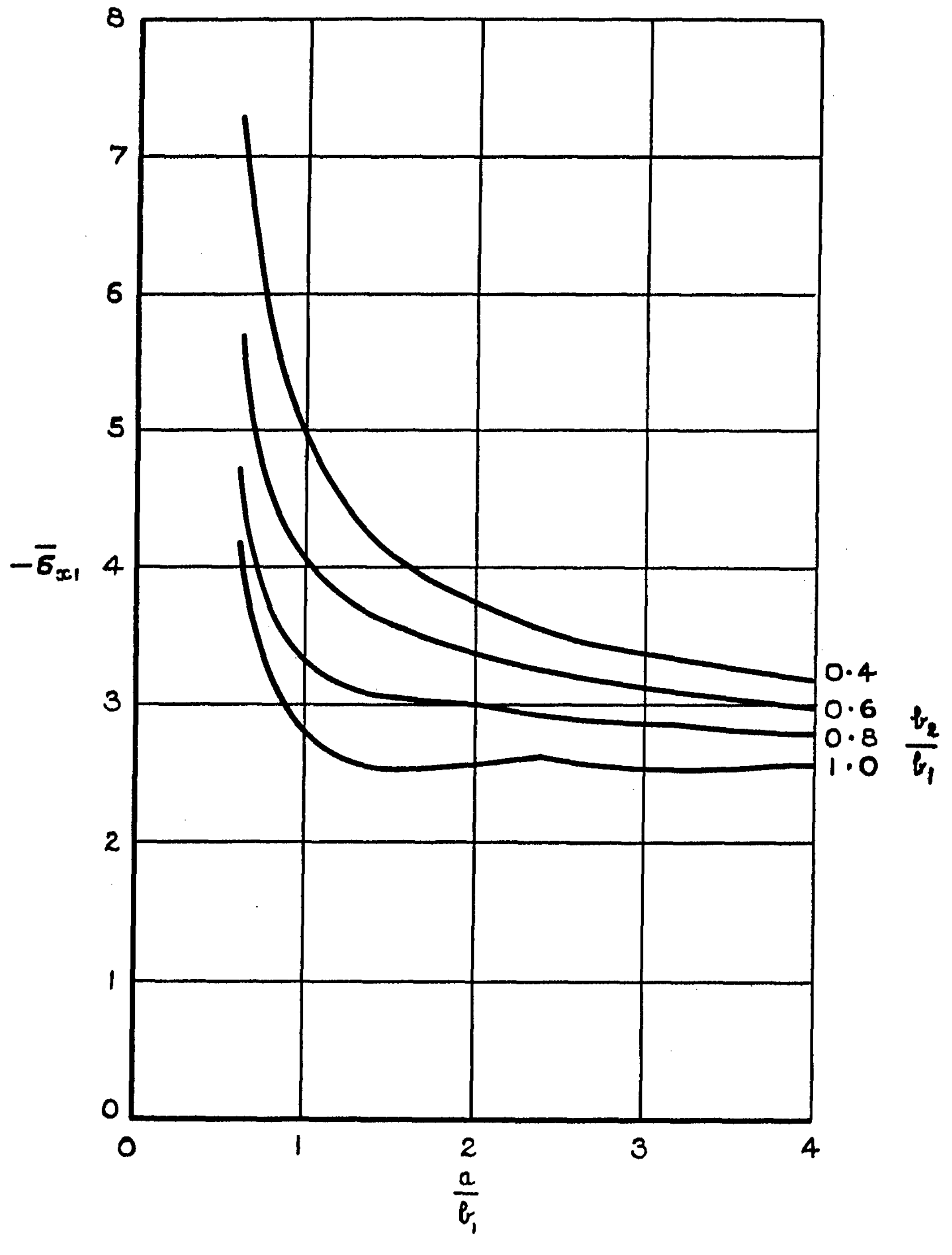


FIG.14. BUCKLING STRESS DIAGRAM. SIDES AND ENDS SIMPLY — SUPPORTED. NO STRAIN NORMAL TO AXIS OF TAPER. $N_{x2}/N_{x1} = 1$.

FIG.15.

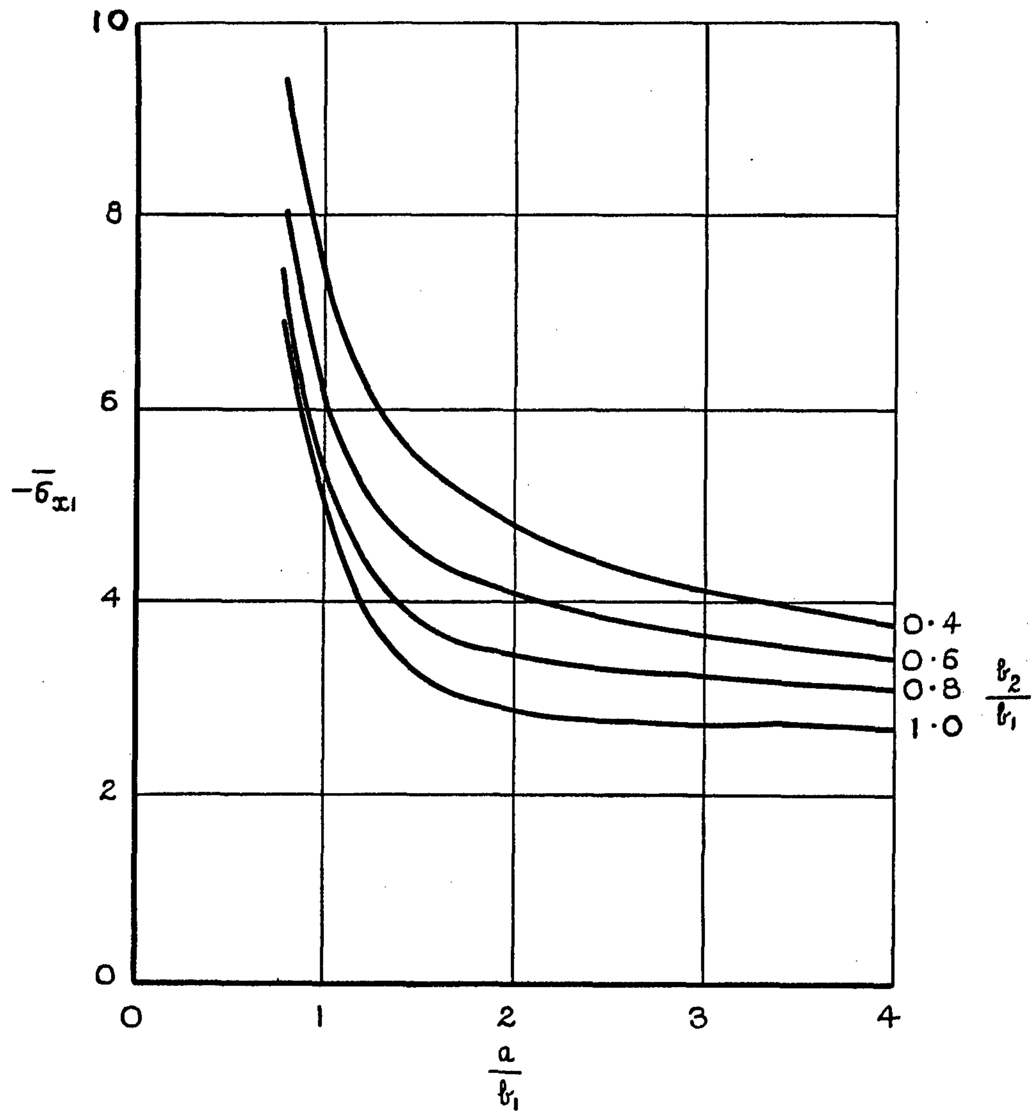


FIG. 15. BUCKLING STRESS DIAGRAM. SIDES SIMPLY-SUPPORTED. ENDS CLAMPED. NO STRAIN NORMAL TO AXIS OF TAPER. $N_{x2}/N_{x1} = 1$.

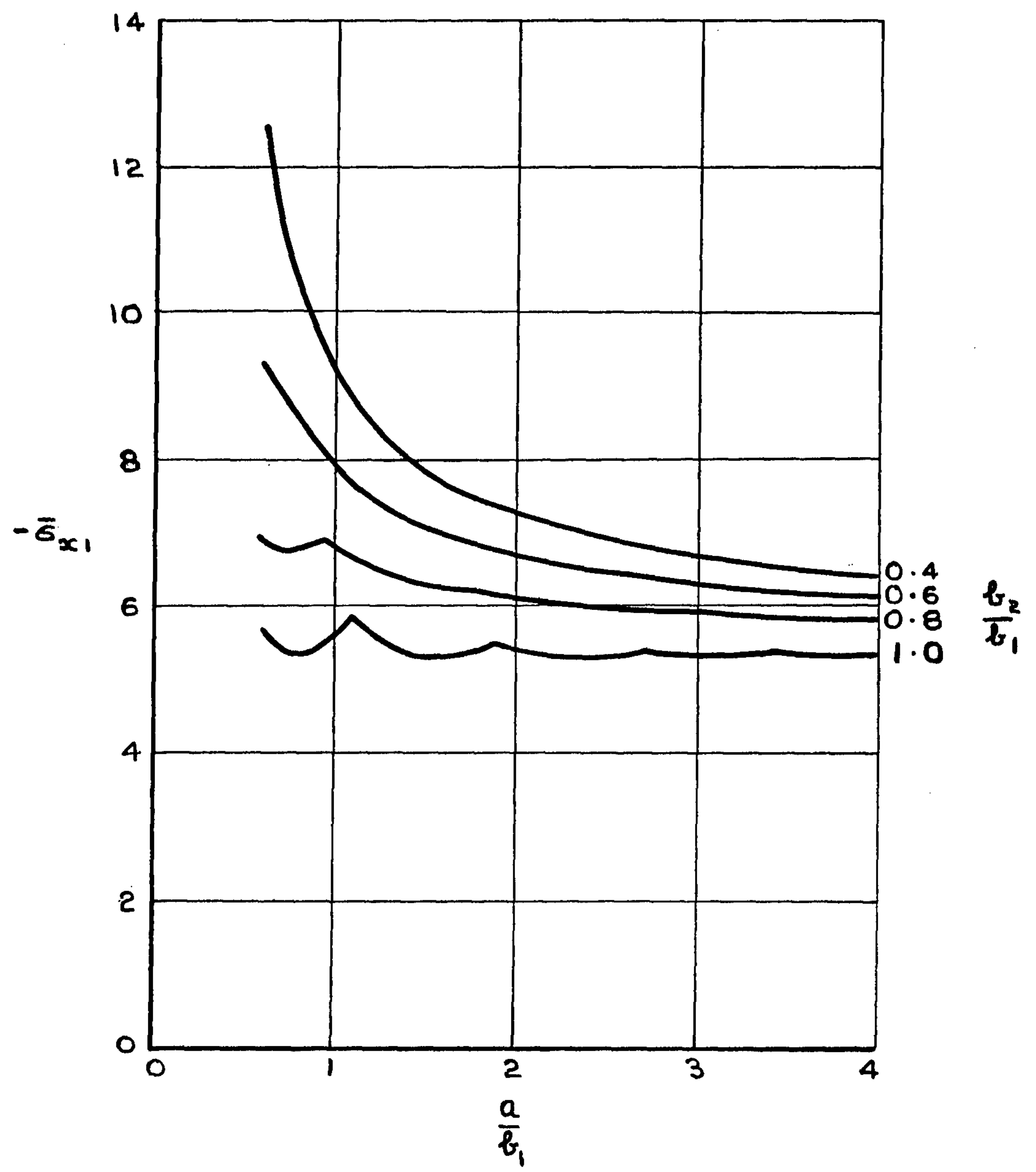


FIG. 16. BUCKLING STRESS DIAGRAM. SIDES CLAMPED. ENDS SIMPLY-SUPPORTED. NO STRAIN NORMAL TO AXIS OF TAPER. $N_{x2}/N_{x1}=1$

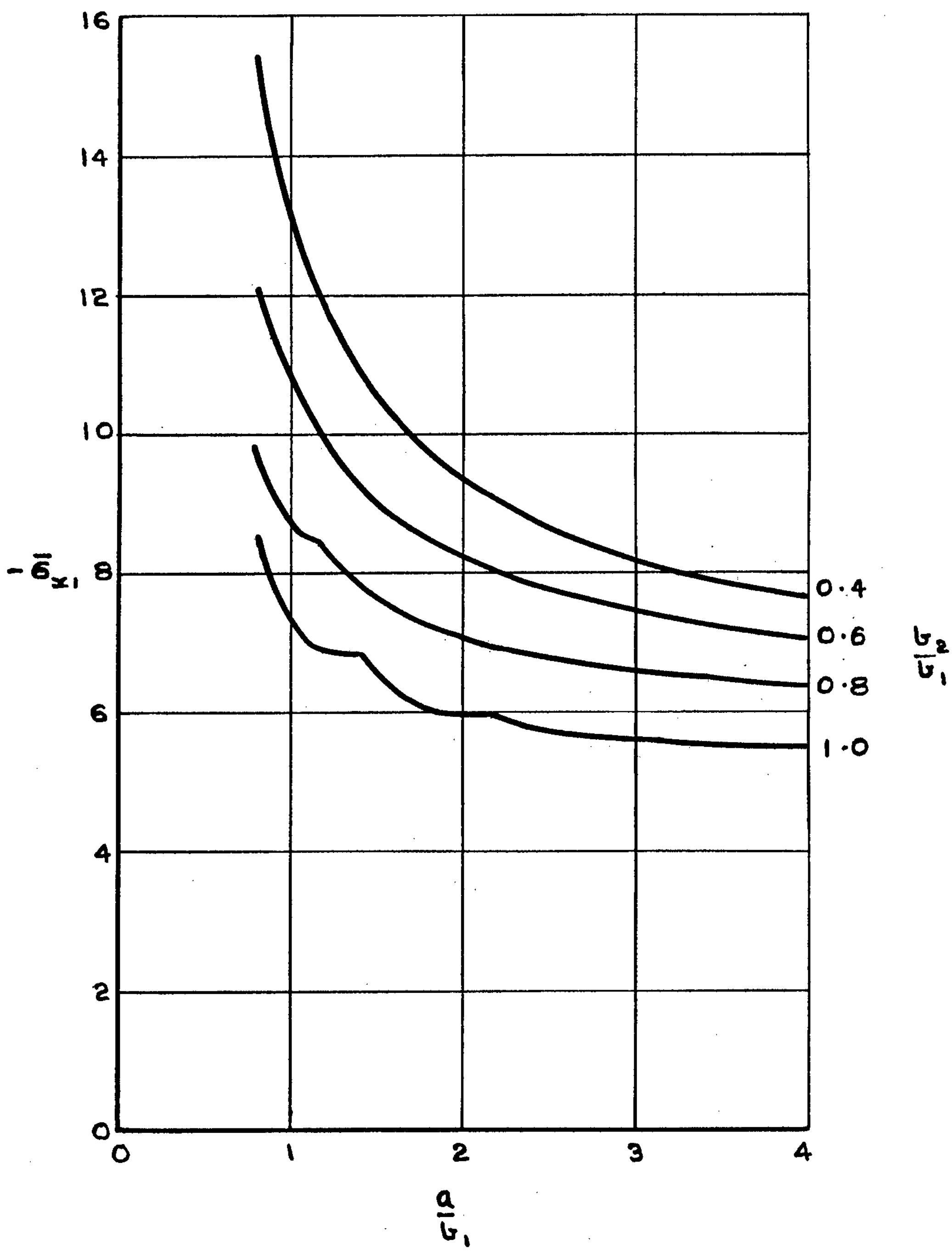


FIG. 17. BUCKLING STRESS DIAGRAM. SIDES AND ENDS CLAMPED. NO STRAIN NORMAL TO AXIS OF TAPER. $N_{x2}/N_{x1} = 1$

FIG. 18.

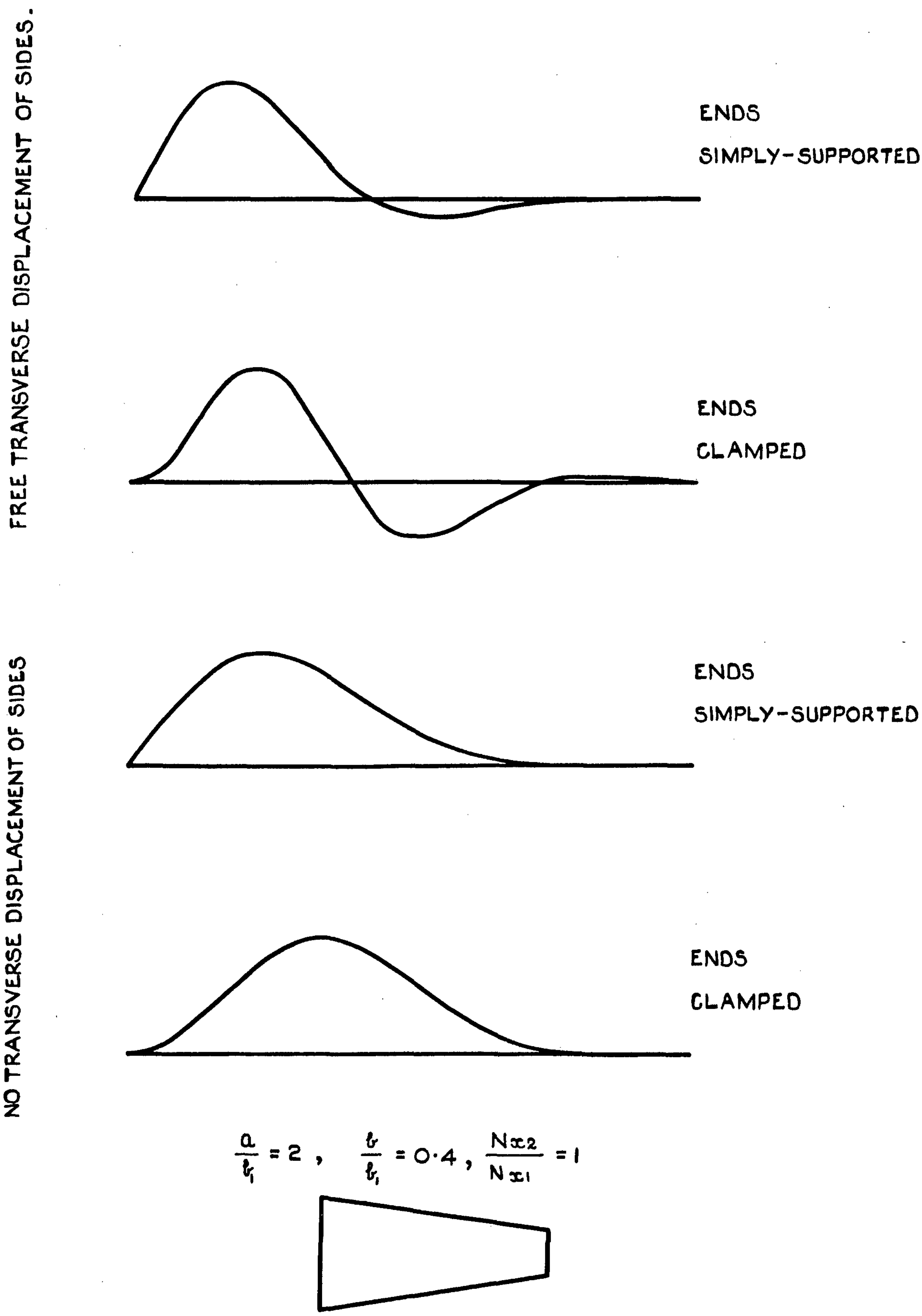
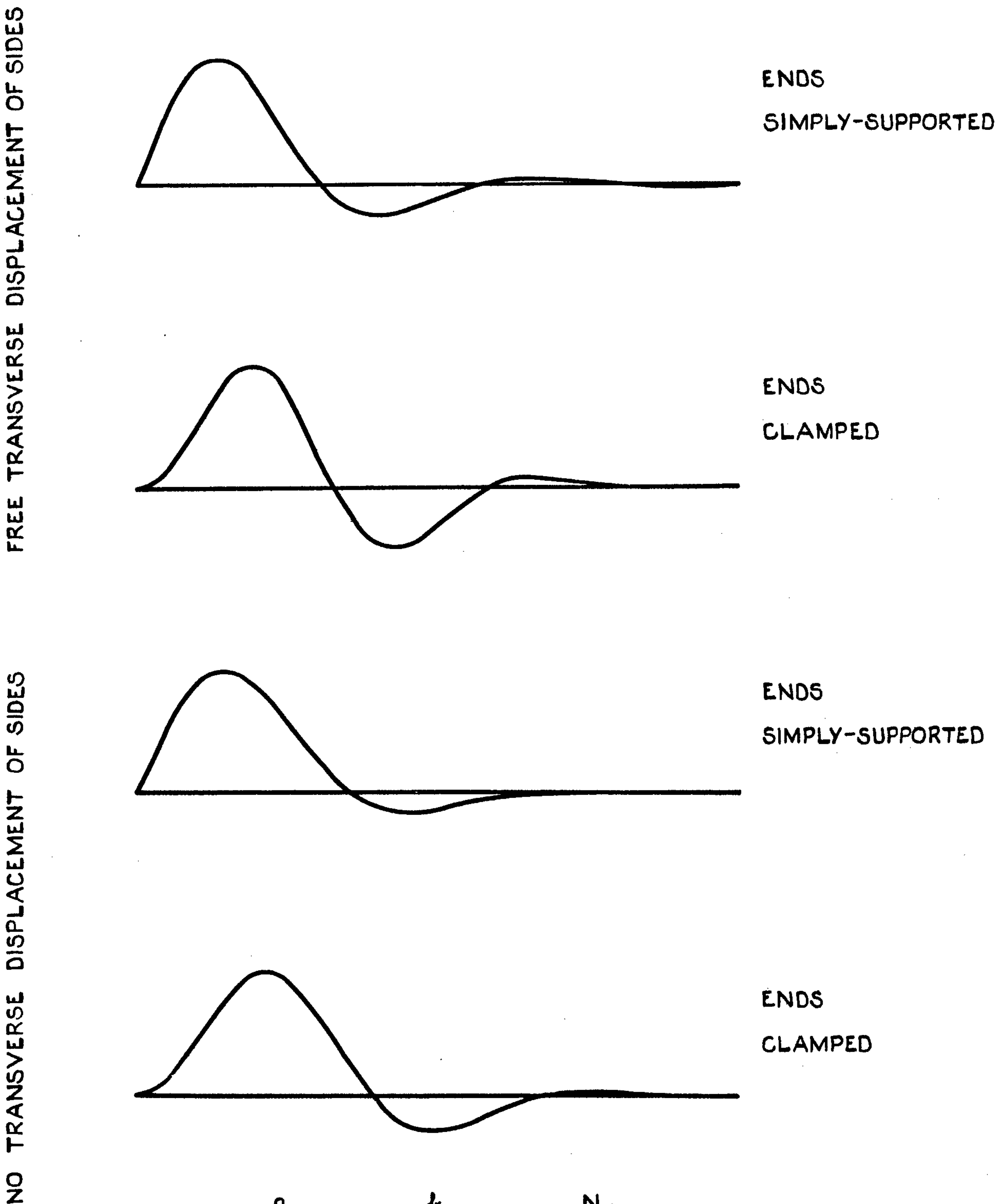


FIG. 18. SPECIMEN BUCKLED SHAPES, SIDES SIMPLY - SUPPORTED.



$$\frac{a}{b_1} = 2 , \quad \frac{b}{b_1} = 0{\cdot}4 , \quad \frac{N_{x2}}{N_{x1}} = 1$$

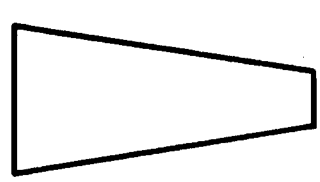


FIG.19. SPECIMEN BUCKLED SHAPES, SIDES CLAMPED.